

Enumerating typical abelian coverings of Cayley graphs

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Abstract

In this article we complete the work of enumerating typical abelian coverings of Cayley graphs, by reducing the problem to enumerating certain subgroups of finite abelian groups.

key words: Cayley graph, typical abelian covering, enumeration, subgroup of abelian group.

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1 Introduction

We consider the problem of enumerating isomorphism classes of typical abelian coverings of Cayley graphs. This problem has recently been considered as one of the central research topics in enumerative topological graph theory and has been partially solved in consecutive papers [6], [7], [8].

First, recall some basic notions in graph theory. One can refer to [9].

Assume all graphs are finite, simple and connected. For a graph G , use $V(G)$ and $E(G)$ to denote the set of vertices and edges, respectively. The neighborhood of a vertex $v \in V(G)$, denoted $N(v)$, is the set of vertices adjacent to v .

A *covering* of graphs $p : \tilde{G} \rightarrow G$ is a surjection $p : V(\tilde{G}) \rightarrow V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \rightarrow N(v)$ is bijective for all $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. Say p is *regular* if $\text{Aut}(\tilde{G})$ acts transitively on each fiber $p^{-1}(v)$. Two coverings

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$p_i : \tilde{G}_i \rightarrow G, i \in \{1, 2\}$, are said to be *isomorphic* if there exists a graph isomorphism $\Phi : \tilde{G}_1 \rightarrow \tilde{G}_2$ such that $p_2 \circ \Phi = p_1$.

Let A be a finite group and let X be a subset of A such that $X = X^{-1}$ and $1 \notin X$. The *Cayley graph on A relative to X* , denoted as $G = \text{Cay}(A, X)$, is the graph having vertex set $V(G) = A$ and edge set $E(G) = \{\{g, gx\} : g \in A, x \in X\}$. It is clear that G is connected if and only if X generates A . A *circulant* graph is a Cayley graph on a cyclic group.

A *typical* covering is a covering $f_* : \text{Cay}(A, X) \rightarrow \text{Cay}(B, Y)$ induced by a surjective group homomorphism $f : A \rightarrow B$ such that $f(X) = Y$; it is regular with covering transformation group $\ker f$. We assume $X \cap \ker f = \emptyset$ and that $f|_X : X \rightarrow Y$ is bijective in order to deal with only simple graphs. The typical covering is called *circulant* (resp. *abelian*) if A is a cyclic (resp. abelian) group.

Typical circulant coverings of a circulant graph were enumerated in [6], [7], and typical abelian coverings with a prime number of folds of a circulant graph were enumerated in [8].

In this paper we get some much more general results. Namely, we count typical abelian coverings of a Cayley graph on any finite abelian group with any given abelian covering transformation group in Theorem 3.6, and count those with any given number of folds in Theorem 3.8; both results are given by explicit formulas. Moreover, quite general but not very concrete results are given in Theorem 3.3 and Theorem 3.4. These are settled by clarifying the connection between typical coverings and subgroups of some abelian groups, and then reducing the problem to enumerating certain subgroups of abelian groups. The topics on enumerations of abelian groups are important and related to ours, so we devote several pages to discussing them in Section 2.

Here are some conventions for notation. For a ring \mathcal{R} , we use $\mathcal{R}^{n,m}$ to denote the set of $n \times m$ matrices with entries in \mathcal{R} and identify $\mathcal{R}^{1,m}$ with \mathcal{R}^m ; let $\text{GL}(m, \mathcal{R})$ denote the set of $m \times m$ invertible matrices with entries in \mathcal{R} . For $M \in \mathcal{R}^{n,m}$, by $\langle M \rangle$ we mean the subgroup of \mathcal{R}^m generated by the row vectors of M . For a finite abelian group A , let $\exp(A)$ denote the exponent of A , that is, the least positive integer s such that $sa = 0$ for each $a \in A$. The cyclic group $\mathbb{Z}/n\mathbb{Z}$ is abbreviated to be \mathbb{Z}_n . For a real number x , the least integer not less than x is denoted as $\lfloor x \rfloor$.

2 Enumerating subgroups of finite abelian groups

2.1 Overview

It is well-known that any finite abelian group A is isomorphic to $\prod_{p \in \Lambda} A_{(p)}$ for some finite set Λ of prime numbers, each $A_{(p)}$ being a p -group. Call $A_{(p)}$ the p -primary part of A .

Each abelian p -group L is isomorphic to $\mathbb{Z}_{p^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p^{\alpha_n}}$ for some partition $\alpha = (\alpha_1, \dots, \alpha_n)$ of $|\alpha| := \sum_{i=1}^n \alpha_i$, with $\alpha_1 \geq \cdots \geq \alpha_n \geq 1$. Call α the *type* of L . Let $\mathcal{A}_p(\alpha)$ denote the collection of abelian p -groups of type α . Regard the trivial group as a p -group of type (0) for any p . Given two partitions $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_m)$, write $\beta \subseteq \alpha$ if $m \leq n$ and $\beta_i \leq \alpha_i$ for $1 \leq i \leq m$.

The problem of counting certain subgroups of a given finite abelian group has a long history. Early in 1948, the number $\mathcal{N}_p(\alpha, \beta)$, for $\beta \subseteq \alpha$, of subgroups of type β of an abelian p -group of type α was determined by Delsarte [4], Djubjuk [5] and Yeh [11]. Suppose $\beta_1 = \cdots = \beta_{m_1}, \beta_{m_1+1} = \cdots = \beta_{m_1+m_2}, \dots, \beta_{m_1+\dots+m_{r-1}+1} = \cdots = \beta_{m_1+\dots+m_r}$ with $m_1 + \cdots + m_r = m$, in which case we generally write

$$\beta = (\beta_{m_1}^{m_1}, \dots, \beta_{m_1+\dots+m_r}^{m_r}) \quad (1)$$

for short, such that $\beta_{m_1} > \beta_{m_1+m_2} > \cdots > \beta_{m_1+\dots+m_r}$, and $\alpha_{\mu_i} < \beta_i \leq \alpha_{\mu_i-1}$ ($i = 1, \dots, m$; set $\alpha_{n+1} = 0$). Then

$$\mathcal{N}_p(\alpha, \beta) = p^H \prod_{i=1}^m (p^{\mu_i-i} - 1) / \prod_{\eta=1}^r \prod_{\nu=1}^{m_\eta} (p^\nu - 1), \quad (2)$$

where

$$H = \sum_{i=1}^m (\mu_i - 2i)(\beta_i - 1) + \frac{1}{2} \left(\sum_{i=1}^r m_i^2 - m^2 \right) + \sum_{i=1}^m \sum_{\eta=\mu_i}^n \alpha_\eta. \quad (3)$$

Remark 2.1. If $\beta_1 \leq \alpha_n$, and $K \in \mathcal{A}_p(\beta)$ is a subgroup of $\prod_{j=1}^n \mathbb{Z}_{p^{\alpha_j}}$, then K must lie in $\{(a_1, \dots, a_n) : p^{\beta_1} a_i = 0, 1 \leq i \leq n\} \cong \mathbb{Z}_{p^{\beta_1}}^n$. This shows

$$\mathcal{N}_p(\alpha, \beta) = \mathcal{N}_p(\beta_1^n, \beta), \quad (4)$$

where we have dropped the parentheses, writing k^n to mean (k^n) .

A refined problem suggested by P. Hall is to determine the number $\mathcal{N}_p(\alpha, \beta, \gamma)$ of subgroups of type β of an abelian p -group of type α which have a quotient group of type γ . A partial result was obtained in [3]. We give a few details on this problem in Section 2.3.

As for another problem, Stehling [10] gave an expression for the number $\mathcal{N}_{p,r}(\alpha)$ of subgroups of order p^r of an abelian p -group of type α , and derived a recurrence relation; the expression is

$$\mathcal{N}_{p,r}(\alpha) = \sum_{\beta \subseteq \alpha: |\beta|=r} \prod_{i=1}^{\alpha_1} \begin{bmatrix} a_i - b_{i+1} \\ b_i - b_{i+1} \end{bmatrix}_p p^{(a_i - b_i)b_{i+1}}, \quad (5)$$

where a_i (resp. b_i) is the number of the integers α_j (resp. β_j) with $\alpha_j \geq i$ (resp. $\beta_j \geq i$), and the Gaussian binomial coefficient is defined as

$$\begin{bmatrix} k \\ l \end{bmatrix}_p = \prod_{i=1}^l \frac{p^{k-l+i} - 1}{p^i - 1}. \quad (6)$$

Also, people paid attention to the total number of all subgroups of an abelian group, see [1], [2].

2.2 On the structure of subgroups of abelian p -groups

Fix a prime number p and a positive integer k .

Definition 2.2. For $\lambda \in \mathbb{Z}_{p^k}$, the p -degree of λ , denoted $\deg_p(\lambda)$, is the unique non-negative integer i such that $\lambda = p^i \cdot \chi$ with χ invertible.

Lemma 2.3. Each subgroup $K \leq \mathbb{Z}_{p^k}^n$ is of the form $\langle PQ \rangle$ such that (i) $P \in \mathbb{Z}_{p^k}^{m,n}$, $m \leq n$, $P_{i,j} = \delta_{i,j} p^{k-\beta_i}$, $k \geq \beta_1 \geq \dots \geq \beta_m \geq 1$; (ii) $Q \in GL(n, \mathbb{Z}_{p^k})$. Moreover, $K \cong \mathbb{Z}_{p^{\beta_1}} \times \dots \times \mathbb{Z}_{p^{\beta_m}}$.

Proof. Denote $R = \mathbb{Z}_{p^k}$.

Take a set of generators M_1, \dots, M_l of K , with $M_i = (M_{i,1}, \dots, M_{i,n})$. Consider the matrix $M \in R^{l,n}$ with the (i, j) -entry $M_{i,j}$. Choose (i_0, j_0) such that $d := \deg_p(M_{i_0, j_0})$ is the smallest among all the $\deg_p(M_{i,j})$'s, and suppose $M_{i_0, j_0} = p^d \cdot \chi$. Take elementary transformations to exchange i_0 -th row with

first row and j_0 -th column with first column of M , and eliminate the $(i, 1)$ -entry for $1 < i \leq l$ and the $(1, j)$ -entry for $1 < j \leq n$, and then divide the first row by χ . The resulting matrix, denoted $M^{(1)}$, is equal to $S_1 M T_1$ for some $S_1 \in \text{GL}(l, R), T_1 \in \text{GL}(n, R)$. Do the same thing to the down-right $(l-1) \times (n-1)$ minor of $M^{(1)}$, and go on. Eventually one obtains a matrix $M^{(m)} = (S_m \cdots S_1) M (T_1 \cdots T_m)$ such that $m \leq n$, $S_i \in \text{GL}(m, R), T_i \in \text{GL}(n, R)$, $M_{i,i}^{(m)} = p^{s_i}, 1 \leq i \leq m$, with $0 \leq s_1 \leq \cdots \leq s_m < k$, and all the other entries vanish. Let $\beta_i = k - s_i$, so that $M^{(m)} = P$. Then $K = \langle M \rangle = \langle PQ \rangle$ with $Q = (T_1 \cdots T_m)^{-1}$.

Since $w \mapsto w \cdot Q$ defines an isomorphism from $\langle P \rangle$ to $\langle PQ \rangle$, we see that $K \cong \langle P \rangle \cong \mathbb{Z}_{p^{\beta_1}} \times \cdots \times \mathbb{Z}_{p^{\beta_m}}$. \square

Corollary 2.4. *If K is a subgroup of type $\beta = (\beta_1, \dots, \beta_m)$ of $\mathbb{Z}_{p^k}^n$, then $\mathbb{Z}_{p^k}^n / K$ has type $k^n - \beta$, where*

$$k^n - \beta = (k^{n-m}, k - \beta_m, \dots, k - \beta_1). \quad (7)$$

Proof. By Lemma 2.3 there exists $Q \in \text{GL}(n, \mathbb{Z}_{p^k})$ such that $K = \langle PQ \rangle$, where $P \in \mathbb{Z}_{p^k}^{m,n}$ with $P_{i,j} = \delta_{i,j} p^{k-\beta_i}$. Since $w \mapsto w \cdot Q$ defines an automorphism of $\mathbb{Z}_{p^k}^n$, we have

$$\mathbb{Z}_{p^k}^n / K \cong \mathbb{Z}_{p^k}^n / \langle P \rangle \cong \mathbb{Z}_{p^k}^{n-m} \times \mathbb{Z}_{p^{k-\beta_m}} \times \cdots \times \mathbb{Z}_{p^{k-\beta_1}} \in \mathcal{A}_p(k^n - \beta). \quad (8)$$

\square

2.3 Subgroups with prescribed quotients

Proposition 2.5. (a) *For any partitions α, β, γ , we have*

$$\mathcal{N}_p(\alpha, \beta, \gamma) = \mathcal{N}_p(\alpha, \gamma, \beta). \quad (9)$$

(b) *For $L \in \mathcal{A}_p(\alpha)$, the number of subgroups K of L such that $L/K \in \mathcal{A}_p(\beta)$ is $\mathcal{N}_p(\alpha, \beta)$.*

Proof. (a) is stated in [2] (at the beginning of page 2).

(b) follows from (a): the number of subgroups K of L such that $L/K \in \mathcal{A}_p(\beta)$ is, summing over the type γ of K ,

$$\sum_{\gamma} \mathcal{N}_p(\alpha, \gamma, \beta) = \sum_{\gamma} \mathcal{N}_p(\alpha, \beta, \gamma) = \mathcal{N}_p(\alpha, \beta).$$

\square

Remark 2.6. *Corollary 2.4 together with Proposition 2.5 (b) shows*

$$\mathcal{N}_p(k^n, \beta) = \mathcal{N}_p(k^n, k^n - \beta). \quad (10)$$

Proposition 2.7. *Suppose $\alpha = (a_1^{n_1}, a_2^{n_2})$ and $s \leq a_1$. Then $\mathcal{N}_p(\alpha, s, \beta)$ takes nonzero values only in the following cases:*

$$\mathcal{N}_p(\alpha, s, \beta) = \frac{p^{n_1} - 1}{p - 1} p^{(s-1)(n_1+n_2-1)}, \quad (11)$$

when $\beta = (a_1^{n_1-1}, a_2^{n_2}, a_1 - s), (a_1^{n_1-1}, a_1 - s, a_2^{n_2})$ or $(a_1^{n_1-1}, a_1 + a_2 - s - b, a_2^{n_2-1}, b)$ for some b with $a_2 - s < b < a_1 - s$;

$$\mathcal{N}_p(\alpha, s, (a_1^{n_1}, a_2^{n_2-1}, a_2 - s)) = \frac{p^{n_2} - 1}{p - 1} p^{(s-1)(n_2-1)+sn_1}. \quad (12)$$

Proof. Let $L = \mathbb{Z}_{p^{a_1}} \times \mathbb{Z}_{p^{a_2}}$.

Suppose

$$u = (u_1^{(1)}, \dots, u_{n_1}^{(1)}, u_1^{(2)}, \dots, u_{n_2}^{(2)}) \in L$$

has order p^s . For $i = 1, 2$, there exists $Q_i \in \text{GL}(n_i, \mathbb{Z}_{p^{a_i}})$ such that

$$(u_1^{(i)}, \dots, u_{n_i}^{(i)})Q_i = (p^{b_i}, 0, \dots, 0).$$

Then $s = \max\{a_1 - b_1, a_2 - b_2\}$. Denote

$$v = (p^{b_1}, 0, \dots, 0, p^{b_2}, 0, \dots, 0).$$

The matrices Q_1, Q_2 define an automorphism of L by

$$(w^{(1)}, w^{(2)}) \mapsto (w^{(1)}Q^{(1)}, w^{(2)}Q^{(2)}),$$

hence $L/\langle u \rangle \cong L/\langle v \rangle$.

There is a canonical injection

$$\iota : \mathbb{Z}_{p^{a_2}} \rightarrow \mathbb{Z}_{p^{a_1}}, \quad \lambda \mapsto p^{a_1-a_2} \cdot \lambda.$$

It can be verified that

$$\begin{aligned} L/\langle v \rangle &\cong \{(w^{(1)}, w^{(2)}) \in L : p^{b_1}w_1^{(1)} + \iota(p^{b_2}w_2^{(2)}) = 0 \text{ in } \mathbb{Z}_{p^{a_1}}\} \\ &\cong \mathbb{Z}_{p^{a_1}}^{n_1-1} \times \mathbb{Z}_{p^{a_2}}^{n_2-1} \times \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{p^{a_1}} : p^{a_2}\lambda_2 = \sum_{i=1}^2 p^{b_i}\lambda_i = 0\}. \end{aligned}$$

(1) When $a_1 - b_1 = s$ and $a_2 - b_2 < s$.

If $b_1 \leq b_2$, then $p^{a_2}\lambda_2 = \sum_{i=1}^2 p^{b_i}\lambda_i = 0$ if and only if $p^{b_1}(\lambda_1 + p^{b_2-b_1}\lambda_2) = p^{a_2}\lambda_2 = 0$, hence

$$L/\langle u \rangle \cong \mathbb{Z}_{p^{a_1}}^{n_1-1} \times \mathbb{Z}_{p^{a_2}}^{n_2} \times \mathbb{Z}_{p^{b_1}}.$$

To count such cyclic subgroups $\langle u \rangle$, note that u can be changed into v with $b_1 = a_1 - s$ and $b_2 > a_2 - s$ by some (Q_1, Q_2) if and only if

$$\begin{aligned} \min\{\deg_p(u_\eta^{(1)}): 1 \leq \eta \leq n_1\} &= a_1 - s, \\ \min\{\deg_p(u_\eta^{(2)}): 1 \leq \eta \leq n_2\} &\geq a_2 - s + 1; \end{aligned}$$

there are $p^{(s-1)(n_1+n_2)}(p^{n_1} - 1)$ choices for u , hence there are

$$\frac{p^{(s-1)(n_1+n_2-1)}(p^{n_1} - 1)}{p^s - p^{s-1}} = \frac{p^{n_1} - 1}{p - 1} p^{(s-1)(n_1+n_2-1)}$$

such cyclic subgroups.

If $b_1 > b_2$, $p^{a_2}\lambda_2 = \sum_{i=1}^2 p^{b_i}\lambda_i = 0$ if and only if $p^{b_2}(p^{b_1-b_2}\lambda_1 + \lambda_2) = p^{a_2+b_1-b_2}\lambda_1 = 0$, hence

$$L/\langle u \rangle \cong \mathbb{Z}_{p^{a_1}}^{n_1-1} \times \mathbb{Z}_{p^{a_2}}^{n_2-1} \times \mathbb{Z}_{p^{b_2}} \times \mathbb{Z}_{p^{a_2+b_1-b_2}},$$

and there are also $\frac{p^{n_1}-1}{p-1} p^{(s-1)(n_1+n_2-1)}$ such subgroups $\langle u \rangle$.

(2) When $a_2 - b_2 = s$, in a similar way one can deduce that

$$L/\langle u \rangle \cong \mathbb{Z}_{p^{a_1}}^{n_1} \times \mathbb{Z}_{p^{a_2}}^{n_2-1} \times \mathbb{Z}_{p^{b_2}},$$

and there are $\frac{p^{n_2}-1}{p-1} p^{(s-1)(n_2-1)+sn_1}$ such subgroups $\langle u \rangle$. \square

3 Enumerating typical abelian coverings of Cayley graphs

In this section, a connection between typical abelian coverings of a Cayley graph on an abelian group and certain subgroups of some abelian group determined by the Cayley graph is established, then the enumeration problem is converted to counting subgroups satisfying some conditions.

The following Lemma was proved in [8]:

Lemma 3.1. *Two connected typical coverings $f_i : \text{Cay}(A_i, X_i) \rightarrow \text{Cay}(B, Y)$, $i = 1, 2$, are isomorphic if and only if there exists a group isomorphism $\phi : A_1 \rightarrow A_2$ such that $f_2 \circ \phi = f_1$ and $\phi(X_1) = X_2$.*

From now on we assume that all groups are abelian.

Given a Cayley graph $\text{Cay}(B, Y)$ such that $Y = -Y$ and $\langle Y \rangle = B$. Suppose $Y = \{\pm y_1, \dots, \pm y_l\} \cup \{y'_1, \dots, y'_{l'}\}$, where $2y_i \neq 0$, $1 \leq i \leq l$, and $2y'_j = 0$, $1 \leq j \leq l'$. Then the valence of $\text{Cay}(B, Y)$ is $2l + l'$. Let

$$Z(Y) = \{(a_1, \dots, a_{l+l'}) \in \mathbb{Z}^{l+l'} : \sum_{i=1}^l a_i y_i + \sum_{j=1}^{l'} a_{l+j} y'_j = 0\}. \quad (13)$$

Then $Z(Y)$ contains the subgroup Z_0 , where

$$Z_0 = \{(a_1, \dots, a_{l+l'}) : a_1 = \dots = a_l = 0; 2|a_{l+j}, 1 \leq j \leq l'\}, \quad (14)$$

and $B \cong \mathbb{Z}^{l+l'} / Z(Y)$.

For $i = 1, \dots, l + l'$, let $E_i = (E_{i,1}, \dots, E_{i,l+l'}) \in \mathbb{Z}^{l+l'}$ with $E_{i,j} = \delta_{i,j}$. Put

$$E = \{\pm E_1, \dots, \pm E_l\} \cup \{E_{l+1}, \dots, E_{l+l'}\}. \quad (15)$$

Lemma 3.2. *Each typical covering $f : \text{Cay}(A, X) \rightarrow \text{Cay}(B, Y)$ with covering transformation group F is isomorphic to a typical covering*

$$\text{Cay}(\mathbb{Z}^{l+l'} / C, q_C(E)) \rightarrow \text{Cay}(\mathbb{Z}^{l+l'} / Z(Y), q_{Z(Y)}(E)) \cong \text{Cay}(B, Y),$$

the covering induced by the canonical epimorphism $\mathbb{Z}^{l+l'} / C \rightarrow \mathbb{Z}^{l+l'} / Z(Y)$, for some C with $Z_0 \leq C \leq Z(Y)$ such that $Z(Y)/C \cong F$, where $q_C : \mathbb{Z}^{l+l'} \rightarrow \mathbb{Z}^{l+l'} / C$ is the canonical quotient map.

Proof. Recall the assumption in Section 1 that $X \cap \ker f = \emptyset$ and $f|_X : X \rightarrow Y$ is bijective. We have $X = \{\pm x_1, \dots, \pm x_l\} \cup \{x'_1, \dots, x'_{l'}\}$, where $x_i = (f|_X)^{-1}(y_i)$, $1 \leq i \leq l$, and $x'_j = (f|_X)^{-1}(y'_j)$, $1 \leq j \leq l'$. Let

$$C = \{(a_1, \dots, a_{l+l'}) \in \mathbb{Z}^{l+l'} : \sum_{i=1}^l a_i x_i + \sum_{j=1}^{l'} a_{l+j} x'_j = 0\}.$$

Clearly $Z_0 \leq C \leq Z(Y)$. The desired isomorphism $A \cong \mathbb{Z}^{l+l'} / C$ is given by sending x_i to $q_C(E_i)$ and sending x'_j to $q_C(E_{l+j})$. \square

Observe that we have a short exact sequence $0 \rightarrow F \rightarrow A \rightarrow B \rightarrow 0$ of finite abelian groups; it follows that $\exp(A) | \exp(B) \exp(F)$. Hence

$$Z := \{(a_1, \dots, a_{l+l'}) \in \mathbb{Z}^{l+l'} : \exp(B) \exp(F) | a_i, 1 \leq i \leq l+l'\} \quad (16)$$

is contained in C . Let

$$\overline{Z(Y)} = Z(Y)/Z, \quad \overline{C} = C/Z, \quad \overline{Z_0} = (Z_0 + Z)/Z. \quad (17)$$

Note that

$$(\mathbb{Z}^{l+l'}/Z)_{(2)} \cong \mathbb{Z}_{2^{k(2)}}^{l+l'}, \quad (18)$$

where $2^{k(2)} = \exp(B_{(2)}) \cdot \exp(F_{(2)})$.

Via this isomorphism $\overline{Z_0} = (\overline{Z_0})_{(2)}$ is identified with

$$\{(w_1, \dots, w_{l+l'}) \in \mathbb{Z}_{2^{k(2)}}^{l+l'} : w_1 = \dots = w_{l'} = 0, \deg_2(w_{l+j}) \geq 1, 1 \leq j \leq l'\} \quad (19)$$

which is isomorphic to $\mathbb{Z}_{2^{k(2)-1}}^{l'}$.

From Lemma 3.1 we see that two typical coverings

$$\text{Cay}(\mathbb{Z}^{l+l'}/C_i, q_{C_i}(E)) \rightarrow \text{Cay}(\mathbb{Z}^{l+l'}/Z(Y), q_{Z(Y)}(E)), \quad i = 1, 2,$$

are isomorphic if and only if $C_1 = C_2$, if and only if $\overline{C_1} = \overline{C_2}$. Thus we have the following classification:

Theorem 3.3. *Given abelian groups A^0 and F , the isomorphism classes of typical abelian coverings $\text{Cay}(A, X) \rightarrow \text{Cay}(B, Y)$ such that $A \cong A^0$ and the covering transformation group is isomorphic to F are in one-to-one correspondence with the subgroups D of $\overline{Z(Y)}$ such that $\overline{Z_0} \leq D$, $(\mathbb{Z}^{l+l'}/Z)/D \cong A^0$ and $\overline{Z(Y)}/D \cong F$.*

Theorem 3.4. *Suppose $B \cong \prod_{p \in \Lambda'} B_{(p)}$ with $B_{(p)} \in \mathcal{A}_p(\alpha(p))$, $F \cong \prod_{p \in \Lambda} F_{(p)}$ with $F_{(p)} \in \mathcal{A}_p(\beta(p))$, and $A^0 \cong \prod_{p \in \Lambda \cup \Lambda'} A_{(p)}^0$ with $A_{(p)}^0 \in \mathcal{A}_p(\gamma(p))$, (Λ, Λ' being finite sets of prime numbers). Then the number of typical abelian coverings $\text{Cay}(A, X) \rightarrow \text{Cay}(B, Y)$ such that $A \cong A^0$ and the covering transformation group is isomorphic to F is*

$$\begin{aligned} & N \cdot \prod_{2 \neq p \in \Lambda - \Lambda'} \mathcal{N}_p(\beta_1(p)^{l+l'}, \beta(p)) \\ & \cdot \prod_{2 \neq p \in \Lambda \cap \Lambda'} \mathcal{N}_p(k(p)^{l+l'} - \alpha(p), k(p)^{l+l'} - \gamma(p), \beta(p)), \end{aligned} \quad (20)$$

where

$$k(p) = \alpha_1(p) + \beta_1(p), \quad (21)$$

and N is the number of subgroups K of type $k(2)^{l+l'} - \gamma(2)$ of $\overline{Z(Y)}_{(2)}$ such that $\overline{Z_0} \leq K$ and $\overline{Z(Y)}_{(2)}/K \in \mathcal{A}_2(\beta(2))$.

Proof. By Theorem 3.3 it is sufficient to count subgroups D of $\overline{Z(Y)}$ with $\overline{Z_0} \leq D$, $(\mathbb{Z}^{l+l'}/Z)/D \cong A^0$ and $\overline{Z(Y)}/D \cong F$.

Write $\overline{Z(Y)} \cong \prod_{p \in \Lambda \cup \Lambda'} \overline{Z(Y)}_{(p)}$ (some $\overline{Z(Y)}_{(p)}$ may be trivial). Formally set $B_{(p)} = 0$ and $\alpha(p) = (0)$ for $p \in \Lambda - \Lambda'$; set $F_{(p)} = 0$ and $\beta(p) = (0)$ for $p \in \Lambda' - \Lambda$. Note that

$$\exp(B) = \prod_{p \in \Lambda'} p^{\alpha_1(p)}, \quad \exp(F) = \prod_{p \in \Lambda} p^{\beta_1(p)}, \quad (22)$$

hence $\mathbb{Z}^{l+l'}/Z \cong \prod_{p \in \Lambda \cup \Lambda'} \mathbb{Z}_{p^{k(p)}}^{l+l'}$.

For each $2 \neq p \in \Lambda \cup \Lambda'$, we have

$$((\mathbb{Z}^{l+l'}/Z)/\overline{Z_0})_{(p)} \cong \mathbb{Z}_{p^{k(p)}}^{l+l'}.$$

By Corollary 2.4,

$$((\mathbb{Z}^{l+l'}/Z)/\overline{Z_0})_{(p)}/(\overline{Z(Y)}/\overline{Z_0})_{(p)} \cong (\mathbb{Z}^{l+l'}/Z)_{(p)}/\overline{Z(Y)}_{(p)} \cong B_{(p)}$$

if and only if $(\overline{Z(Y)}/\overline{Z_0})_{(p)} \in \mathcal{A}_p(k(p)^{l+l'} - \alpha(p))$;

$$((\mathbb{Z}^{l+l'}/Z)/\overline{Z_0})_{(p)}/(D/\overline{Z_0})_p \cong A_{(p)}^0$$

if and only if $(D/\overline{Z_0})_{(p)} \in \mathcal{A}_p(k(p)^{l+l'} - \gamma(p))$.

Thus the number of subgroups K of $\overline{Z(Y)}_{(p)}$ with $(\overline{Z_0})_{(p)} \leq K$, $\mathbb{Z}_{p^{k(p)}}^{l+l'}/K \cong A_{(p)}^0$ and $\overline{Z(Y)}_{(p)}/K \cong F_{(p)}$ is

$$\mathcal{N}_p(k(p)^{l+l'} - \alpha(p), k(p)^{l+l'} - \gamma(p), \beta(p)).$$

Note that if $p \in \Lambda' - \Lambda$, then $\beta(p) = 0$, hence

$$\mathcal{N}_p(k(p)^{l+l'} - \alpha(p), k(p)^{l+l'} - \gamma(p), \beta(p)) = 1;$$

if $p \in \Lambda - \Lambda'$, then $\alpha(p) = 0, \gamma(p) = \beta(p)$, hence

$$\begin{aligned} & \mathcal{N}_p(k(p)^{l+l'} - \alpha(p), k(p)^{l+l'} - \gamma(p), \beta(p)) \\ &= \mathcal{N}_p(\beta_1(p)^{l+l'}, \beta_1(p)^{l+l'} - \beta(p), \beta(p)) \\ &= \mathcal{N}_p(\beta_1(p)^{l+l'}, \beta(p)) \quad (\text{by Corollary 2.4}). \end{aligned}$$

The case of $p = 2$ can be dealt with similarly.

The result is established. \square

Remark 3.5. Suppose A^0 is cyclic, so that $F \cong \prod_{p \in \Lambda} \mathbb{Z}_{p^{\beta_1(p)}}, B \cong \prod_{p \in \Lambda'} \mathbb{Z}_{p^{\alpha_1(p)}}$, then $\gamma(p) = k(p)$. Suppose the valence of $\text{Cay}(B, Y)$ is d .

If d is odd, then $l' = 1, \alpha_1(2) \geq 1$, and

$$\overline{Z}_0 = \{w_1, \dots, w_{l+1} \in \mathbb{Z}_{2^{k(2)}}^{l+1} : w_1 = \dots = w_l = 0, \deg_2(w_{l+1}) \geq 1\}.$$

If $K \leq \overline{Z(Y)}_{(2)}$ has type $k(2)^{l+l'} - \gamma(2) = k(2)^l$ and contains \overline{Z}_0 , then $(0, \dots, 0, 2) = 2w$ for some $w = (w_1, \dots, w_{l+1}) \in K$. We have

$$\deg_2(w_{l+1}) = 0; \quad \deg_2(w_i) \geq k(2) - 1, 1 \leq i \leq l.$$

When $\beta_1(2) \geq 1$, it is impossible that $w \in \overline{Z(Y)}_{(2)}$, so $N = 0$; when $\beta_1(2) = 0$, it is clear that $N = 1$.

If d is even, then $l' = 0$, and $N = \mathcal{N}_2(k(2)^{l+l'} - \alpha(2), k(2)^{l+l'} - \gamma(2), \beta(2))$.

For $p \in \Lambda \cap \Lambda'$,

$$\begin{aligned} & \mathcal{N}_p(k(p)^{l+l'} - \alpha(p), k(p)^{l+l'} - \gamma(p), \beta(p)) \\ &= \mathcal{N}_p(k(p)^{l+l'} - \alpha_1(p), k(p)^{l+l'-1}, \beta_1(p)) \\ &= \mathcal{N}_p(k(p)^{l+l'} - \alpha_1(p), \beta_1(p), k(p)^{l+l'-1}) \quad (\text{by Proposition 2.5(a)}) \\ &= p^{\beta_1(p)(l+l'-1)} \quad (\text{by (12)}); \end{aligned}$$

for $p \in \Lambda - \Lambda'$,

$$\mathcal{N}_p(k(p)^{l+l'}, k(p)) = p^{(\beta_1(p)-1)(l+l'-1)}(p^{l+l'} - 1).$$

Summarizing, the number of $(\prod_{p \in \Lambda} p^{\beta_1(p)})$ -fold typical circulant coverings of

$$\text{Cay}(B, Y) \text{ is } \begin{cases} 0, & d \text{ is odd and } \beta_1(2) \geq 1 \\ \prod_{p \in \Lambda} N(p), & \text{otherwise,} \end{cases} \quad \text{with}$$

$$N(p) = \begin{cases} p^{\beta_1(p)(\lfloor \frac{d}{2} \rfloor - 1)}, & p \in \Lambda', \\ p^{(\beta_1(p)-1)(\lfloor \frac{d}{2} \rfloor - 1)}(p^{\lfloor \frac{d}{2} \rfloor} - 1), & p \notin \Lambda'. \end{cases} \quad (23)$$

This recovers the main result of [7] (Theorem 13).

Theorem 3.6. Suppose $F \cong \prod_{p \in \Lambda} F_{(p)}$, where Λ is a finite set of prime numbers, and $F_{(p)} \in \mathcal{A}_p(\beta(p))$, for all $p \in \Lambda$. Suppose

$$\beta(p) = (\beta_1(p), \dots, \beta_{m(p)}(p)) = (\bar{\beta}_1(p)^{m_1(p)}, \dots, \bar{\beta}_{r(p)}(p)^{m_{r(p)}(p)}), \quad (24)$$

such that $\bar{\beta}_1(p) > \dots > \bar{\beta}_{r(p)}(p)$. Then the number of typical coverings of $\text{Cay}(B, Y)$ with covering transformation group isomorphic to F is

$$2^{H'(2)} \cdot \prod_{i=1}^{m'(2)} (2^{l+1-i} - 1) \cdot \prod_{j=m'(2)+1}^{m(2)} (2^{l+l_0+1-j} - 1) \cdot \prod_{t=1}^{r(2)} \prod_{\nu=1}^{m_t(2)} (2^\nu - 1)^{-1} \\ \cdot \prod_{2 \neq p \in \Lambda} \left(p^{H(p)} \cdot \prod_{i=1}^{m(p)} (p^{l+l'+1-i} - 1) / \left(\prod_{t=1}^{r(p)} \prod_{\nu=1}^{m_t(p)} (p^\nu - 1) \right) \right), \quad (25)$$

where l_0 is the rank of $Z'(Y)$, with

$$Z'(Y) = \{(u_1, \dots, u_{l'}) \in \mathbb{Z}_2^{l'} : \sum_{j=1}^{l'} u_j y'_j = 0\}, \quad (26)$$

$m'(2)$ is the largest η such that $\beta_\eta(2) > 1$,

$$H'(2) = \sum_{i=1}^{m'(2)} (l+1-2i)(\beta_i(2) - 1) + \frac{1}{2} \left(\sum_{i=1}^{r(2)} m_i(2)^2 - m(2)^2 \right) + l_0 m'(2), \quad (27)$$

and for $p \neq 2$,

$$H(p) = \sum_{i=1}^{m(p)} (l+l'+1-2i)(\beta_i(p) - 1) + \frac{1}{2} \left(\sum_{i=1}^{r(p)} m_i(p)^2 - m(p)^2 \right). \quad (28)$$

Proof. For $p \neq 2$, by Theorem 3.4 it suffices to compute

$$\sum_{\gamma(p)} \mathcal{N}_p(k(p)^{l+l'} - \alpha(p), k(p)^{l+l'} - \gamma(p), \beta(p)) = \mathcal{N}_p(k(p)^{l+l'} - \alpha(p), \beta(p)),$$

which is equal to $\mathcal{N}_p(\beta_1(p)^{l+l'}, \beta(p))$ by Remark 2.1. Then apply (2).

For $p = 2$, summing over $\gamma(2)$ the number $N = N(\gamma(2))$ of the subgroups K of type $k(2)^{l+l'} - \gamma(2)$ of $\overline{Z(Y)}_{(2)}$ such that $\overline{Z_0} \leq K$ and $\overline{Z(Y)}_{(2)}/K \in \mathcal{A}_2(\beta(2))$ and applying Proposition 2.5 (b), we obtain $\mathcal{N}_2(\tau, \beta(2))$, where τ is the type of $\overline{Z(Y)}_{(2)}/\overline{Z_0}$.

Now if $K' \leq \mathbb{Z}_{2^{k(2)}}^{l+l'}/\overline{Z_0} \cong \mathbb{Z}_{2^{k(2)}}^l \times \mathbb{Z}_2^{l'}$ has type $\beta(2)$, and $w = (w_1, \dots, w_{l+l'}) \in K'$, with $w_1, \dots, w_l \in \mathbb{Z}_{2^{k(2)}}^l$ and $w_{l+1}, \dots, w_{l+l'} \in \mathbb{Z}_2$, then

$$\deg_2(w_i) \geq \alpha_1(2), 1 \leq i \leq l,$$

so $(w_1, \dots, w_l, 0, \dots, 0) \in \overline{Z(Y)}_{(2)}/\overline{Z_0}$. Hence $w \in \overline{Z(Y)}_{(2)}/\overline{Z_0}$ if and only if $(w_{l+1}, \dots, w_{l+l'}) \in Z'(Y)$. This shows

$$\mathcal{N}_2(\tau, \beta(2)) = \mathcal{N}_2((k(2)^l, 1^{l_0}), \beta(2)),$$

which is equal to $\mathcal{N}_2((\beta_1(2)^l, 1^{l_0}), \beta(2))$, by an argument similar to that in Remark 2.1. Then apply (2). \square

Remark 3.7. Consider the special case when F is cyclic: $F \cong \mathbb{Z}_f$ with $f = \prod_{p \in \Lambda} p^{\beta_1(p)}$. Then (25) becomes

$$N'(2) \cdot \prod_{2 \neq p \in \Lambda} \left(p^{(l+l'-1)(\beta_1(p)-1)} \cdot \frac{p^{l+l'} - 1}{p - 1} \right), \quad (29)$$

where $N'(2) = 1$ if $2 \notin \Lambda$, and $N'(2) = \begin{cases} 2^{(l-1)(\beta_1(2)-1)+l_0}(2^l - 1), & \beta_1(2) > 1 \\ 2^{l+l_0} - 1, & \beta_1(2) = 1 \end{cases}$ if $2 \in \Lambda$. If f is prime and B is cyclic, then $l' \in \{0, 1\}$ and the result specializes to Theorem 1 of [8].

Theorem 3.8. Suppose $q = \prod_{p \in \Lambda} p^{s(p)}$. The number of q -fold typical abelian coverings of $\text{Cay}(B, Y)$ is

$$\left(\sum_{k=\lfloor \frac{s(2)-l_0}{l} \rfloor}^{s(2)} \sum_{\substack{0 \leq b_k \leq \dots \leq b_1 \leq l+l_0 \\ b_1 + \dots + b_k = s(2)}} \begin{bmatrix} l + l_0 - b_2 \\ b_1 - b_2 \end{bmatrix}_2 2^{(l+l_0-b_1)b_2} \cdot \prod_{i=2}^k \begin{bmatrix} l - b_{i+1} \\ b_i - b_{i+1} \end{bmatrix}_2 2^{(l-b_i)b_{i+1}} \right) \\ \cdot \prod_{2 \neq p \in \Lambda} \left(\sum_{k=\lfloor \frac{s(p)}{l+l'} \rfloor}^{s(p)} \sum_{\substack{0 \leq b_k \leq \dots \leq b_1 \leq l+l' \\ b_1 + \dots + b_k = s(p)}} \prod_{i=1}^k \begin{bmatrix} l + l' - b_{i+1} \\ b_i - b_{i+1} \end{bmatrix}_p p^{(l+l'-b_i)b_{i+1}} \right). \quad (30)$$

Proof. From Theorem 3.6 and its proof we see that it is enough to compute

$$\sum_{\beta(2) \leq (s(2)^l, 1^{l_0}), |\beta(2)|=s(2)} \mathcal{N}_2((\beta_1(2)^l, 1^{l_0}), \beta(2)) = \sum_{\beta_1(2) = \lfloor \frac{s(2)-l_0}{l} \rfloor}^{s(2)} \mathcal{N}_{2,s(2)}((\beta_1(2)^l, 1^{l_0})),$$

and

$$\sum_{\beta(p) \leq s(p)^{l+l'}, |\beta(p)|=s(p)} \mathcal{N}_p(\beta_1(p)^{l+l'}, \beta(p)) = \sum_{\beta_1(p) = \lfloor \frac{s(p)}{l+l'} \rfloor}^{s(p)} \mathcal{N}_{p,s(p)}(\beta_1(p)^l).$$

The result is proved by applying the formula (5). \square

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